The Semi-Polish Theorem:
one-sided vs joint continuity in groups

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Abstract

We give a symmetrized, unilaterally generated topological condition which ensures that a right-topological group is a Polish topological group.

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1 Introduction

In an algebraic group, when does one-sided continuity of multiplication imply its joint continuity and further its *admissibility,* i.e. endowment of a topological group structure? This question was considered in the *abelian* context by Ellis in [Ell1] (see in particular his Th. 2, where the topology is locally compact – cf. Section 4 below), but otherwise the existing literature, which goes back to Montgomery [Mont2] and also Ellis [Ell2] via Namioka [Nam], considers some form of weak bilateral continuity, usually separate continuity, supported by additional topological features, including some form of completeness. See Bouziad's two papers [Bou1] and [Bou1] for the state-of-the-art results, deducing automatic joint continuity from separate continuity (and for a review of the historic literature), and the more recent paper of Solecki and Srivastava [SolSri], where separate continuity is weakened. For the broader context of automatic continuity see [THJ] (e.g. p. 338) and for the interaction of topology and algebra see Dales [Dal].

By contrast to these bilateral conditions, in the Main Theorem below we assume only a particular form of one-sided continuity, supported by additional topological properties. A contribution of this paper is to replace the use of local compactness (or even subcompactness, for which see [Bou1]) by the recently isolated much weaker notion of shift-compactness in groups given here in the analytic format of Theorem IV in Section 3 (cf. [BOst-N]) and studied for its relationship to analyticity (definition below) in the companion paper [Ost-LBIII], results required from there being identified in Section 3.

A *right-topological group* X with neutral element e is an algebraic group with a topology under which the right-shifts $\rho_b(x) : x \mapsto xb$ are continuous, and so homeomorphisms. Say that a metric d_R^X on X is (uniformly) *compatible* with the algebraic and topological structures on X if it is *right-invariant*, i.e. $d_R^X(xa, ya) = d_R^X(x, y)$, and generates the topology. The metric topology of d_R^X makes the right-shifts bi-uniformly continuous homeomorphisms (i.e. both ρ_b and $\rho_b^{-1} = \rho_{b^{-1}}$ are uniformly continuous). A compatible metric exists (see e.g. by [SeKu] Th. 7.3.1) iff there exits a metric for which the right-shifts are uniformly continuous.

The right-invariant metric d_R^X is retrievable from the function $||x|| := d_R^X(x, e)$ via the formula $d_R^X(x, y) := ||xy^{-1}||$, provided that the function $||x||$ obeys the following *group-norm* axioms.

Definition. For T an *algebraic* group with neutral element e, say that $|| \cdot || : T \to \mathbb{R}_+$ is a *group-norm* ([BOst-N]) if the following properties hold:
(i) *Subadditivity* (Triangle inequality): $||st|| \leq ||s|| + ||t||$;
(ii) *Positivity*: $||t|| > 0$ for $t \neq e$ and $||e|| = 0$;
(iii) *Inversion* (Symmetry): $||t^{-1}|| = ||t||$.
Then $(T, ||.||)$ is a *normed group.*

The conjugate left-topological group structure on X is obtained by taking $d_L^X(x, y) := d_R^X(x^{-1}, y^{-1})$. This is a left-invariant metric on X under which the left-shifts $\lambda_a(x) : x \mapsto ax$ are bi-uniformly continuous. Note that $||x|| = d_L^X(x, e)$. That is, both metrics generate the same norm. So henceforth we will refer to X as a *normed group.* (See [BOst-N] for background and references, and for examples, drawn from groups of self-homeomorphisms of a metric space, see §4.2 and the companion paper [Ost-LBIII].) Here we are concerned with the join of the two metric topologies (coarsest joint refinement), which is generated by the symmetrized metric

$$d_S^X := \max\{d_R^X, d_L^X\}.$$

One has $||x|| = d_S^X(x, e)$, i.e. d_S^X also defines the same norm, which emphasizes that the symmetrized topology is imposed by the assumed one-sided structure. It is now natural to study topological assumptions on d_S^X. To state succinctly our main automatic continuity result (of interest only for the non-abelian context) we need the definition below.

Definitions. For any topological property P, say that the normed group X has the *symmetrized-P property*, or more briefly: X is *semi-P,* if (X, d_S^X)

has property P. In particular, call a normed group X *semi-Polish* if it is *symmetrized-Polish*, i.e., (X, d_S) is topologically complete and separable. (This name was suggested by Anatole Beck.)

Recall that, in a Hausdorff space, a set is *analytic* if it is the continuous image of a Polish space (a separable metric space which is topologically complete) – see [Jay-Rog] for details. So say that the normed group X is *semi-analytic* if X is analytic as a subset of the space (X, d_S).

We freely use the fact that an analytic set has the Baire property (cf. [Kech] Th. 21.6, the Lusin-Sierpiński Theorem, and the closely related Cor. 29.14, Nikodym Theorem, cf. the treatment in [Kur-1] Cor. 1 p. 482 or [Jay-Rog] pp. 42-43). We refer to this result as the *Lusin-Sierpiński-Nikodym Theorem*, abbreviated to LSN. Our interest in analyticity as carrier of the Baire property was motivated by van Mill's proof in [vM] of an analytic form of the Effros Theorem, which de facto assumes only a normed group context.

Our main result is the following.

Theorem 1 ***(Main Theorem: Semi-Polish Theorem).*** *For a normed group X under d_R^X, if the space X is non-meagre and semi-Polish (more generally, semi-analytic), then it is a Polish topological group (i.e. under the d_R^X topology X is completely metrizable and a topological group).*

Of course a metrizable topological group has a right-invariant metric by the Birkhoff-Kakutani Normability Theorem ([Bir], [Kak], cf. [Ost-LBIII]), and a Polish group is non-meagre (Baire's Theorem), so this theorem covers all Polish groups.

The theorem also generalizes a result due to [Loy] and [HJ, Th. 2.3.6 p. 355] that a Baire analytic *topological* group is Polish, granted that an analytic group is separable and metrizable (for which see [HJ, Th. 2.3.6 p. 355]).

We have shown elsewhere ([BOst-N], [Ost-LBIII]) that a modicum of comparability between the left and right norm topologies implies that they are equal and admissible, i.e. that (X, d_R^X) is a topological group; a convenient list may be found in §4.2. The semi-Polish theorem is thus yet another example of this phenomenon.

The rest of the paper is organized as follows. In Section 2 we consider some further results derived from the assumption of symmetrized properties and prove the main theorem. This relies on some results obtained in the companion paper [Ost-LBIII], so for self-sufficiency these are itemized in Section 3 as Theorems I-IV. The concluding remarks in Section 4 comment on the significance of normed groups and why they are either topological or pathological. The non-separable variant of Theorem 1 is briefly discussed; see [Ost-AB] for details.

Notation. We use the subscripts R, L, S as in $x_n \to_R x$ etc. to indicate convergence in the corresponding metrics d_R, d_L, d_S derived from the norm (so that e.g. $d_R(x, y) := ||xy^{-1}||$).

2 Symmetrized properties

For X a normed group, recall the symmetrization metric $d_S^X := \max\{d_R^X, d_L^X\}$ of §1. Its significance comes from the theorem that, for (T, d^T) any complete metric space, the group of bounded self-homeomorphisms of T is complete under the symmetrization of the supremum metric (for details see [Ost-LBIII] and [Dug], Th. XIV.2.6, p. 296). In this section we study d_S as the common refinement of the left and right metrics. This is a natural tool of comparison, as both are 'co-topologies' of d_S – recall from [AdGMcD1] that a topology $\widehat{\mathcal{T}}$ coarser than a regular topology $\mathcal{T}$ (i.e. with $\widehat{\mathcal{T}} \subseteq \mathcal{T}$) is a *co-topology* for $\mathcal{T}$ if $\mathcal{T}$ has a neighbourhood base consisting of $\widehat{\mathcal{T}}$-closed sets. We are unaware of any similar analysis in the literature, save for the work of Itzkowitz and his collaborators: see e.g. [IRSW] for a different analysis, conducted in the broader category of uniform spaces, which compares left and right uniformities. We will be guided here by the result of [BOst-N] (Th.3.9 – Ambidextrous refinement) that (X, d_S) is a topological group iff (X, d_R) is a topological group. A further application to the group of bounded self-homeomorphisms and the subgroup of bi-uniform homeomorphisms, which is a complete topological group under the symmetrized supremum metric is described in §4.1.

Definitions. 1. We recall that a set is *precompact* (or relatively compact, [Dug] XI.6) if its closure is compact, and that a metric $d = d^X$ on X is *proper* if all the closed balls $\bar{B}_d(x, r) := \{y : d(x, y) \leq r\}$ are compact, i.e. the metric has the *Heine-Borel* property: closed and bounded is equivalent to compact. (In geodesic geometry a proper metric space is called 'finitely compact', since an infinite bounded set has a point of accumulation – see [Bus2], or [BH] for a more recent text-book account of the extensive use of this concept.)

2. Say that the group-norm $||.||$ on X is *right* (resp. *left*) *proper* if

d_R^X (resp. d_L^X) is a proper metric, i.e. norm-bounded sets are *precompact*, equivalently closed balls are compact.

3. Say that a group-norm $||.||$ is *proper* if it is either right-proper or left-proper.

Lemma 1 below serves to motivate the property appearing in Theorem 2 (ii) below, and will be used later. The property is of interest, as it requires less than continuity of inversion. For proof see the literature cited.

Lemma 1 (cf. [BePe] IV Th. 1.1). *For self-homeomorphisms g, h, h_n of a metric space (T, d^T), if $h(t) := \lim_n h_n(t)$ and $g(t) := \lim_n h_n^{-1}(t)$ uniformly, then $h \circ g$ is the identity: $h(g(t)) = t$ for all t.*

Theorem 2 (Proper Symmetrization Theorem). *For X a normed group and $d_S := \max\{d_R, d_L\}$,*
(i) if (X, d_S) is a proper metric space, then (X, d_R) is a proper metric space homeomorphic to (X, d_S) under the embedding map $j : (X, d_S^X) \to (X, d_R^X)$ with $j(x) = x$, so is topologically complete; in particular, if (X, d_S) is compact, then (X, d_R) is homeomorphic to (X, d_S);
(ii) conversely, d_S^X is proper for d_R^X proper provided that:
if $x_n \to_R x$ and $x_n^{-1} \to_R y$, then $y = x^{-1}$.
Proof. Note that the embedding $j : (X, d_S^X) \to (X, d_R^X)$ with $j(x) = x$ is continuous, so in particular for (X, d_S^X) Polish (X, d_R) is analytic (in fact absolutely Borel – see closing comments). As d_R is right-invariant, we have $d_R(t^{-1}, e_T) = d_R(e_T, t)$ and so $d_S(t, e_T) = d_R(t, e) = ||t||$. So if d_S is a proper metric, then the norm is proper under d_S.
(i) Suppose that d_S is proper. Then j is closed; otherwise there is a d_S-closed set F that is not d_R-closed, and so there is a sequence x_n in F with d_R-limit $y \notin F$. Being convergent, $||x_n||$ is bounded. As d_S is proper, w.l.o.g. we may assume that x_n is convergent, with d_S-limit x say. So x is in F, as F is d_S-closed. But $d_R(x_n, x) \to 0$, and so $x = y \notin F$, a contradiction. As j is closed and a bijection, it is also open, and so a homeomorphism between (X, d_S) and (X, d_R). As the central balls $\bar{B}_r := \{x : ||x|| \leq r\}$ are sequentially compact under d_S, they are sequentially compact, and so compact, under d_R; so d_R is proper. (Likewise d_L is proper.)
Now suppose only that (X, d_S) is compact. But then d_S is proper, and so (X, d_R) is homeomorphic to (X, d_S).
(ii) If d_R^X is proper, then d_L^X is proper. We show that each $\bar{B}_r$ is compact

under d_S. Indeed, if $||x_n||$ is bounded, then there is an increasing sequence $r(n)$ of integers with $x_{r(n)}$ converging under d_R, i.e. $d_R(x_{r(n)}, x) \to 0$, for some x. Again as $||x_{r(n)}||$ is bounded, there is a subsequence $m(n)$ of $r(n)$ with $x_{m(n)}$ converging under d_L, i.e. $d_R(x^{-1}_{m(n)}, y) \to 0$ for some y. Then $x_{m(n)}$ converges to x and $x^{-1}_{m(n)}$ converges to y, so $y = x^{-1}$. So $d_L(x_{m(n)}, x) \to 0$ as well as $d_R(x_{m(n)}, x) \to 0$, and so $d_S(x_{m(n)}, x) \to 0$. ■

Preliminaries for the proof of Theorem 1. We will need two lemmas. The first is a sharpening appropriate for normed groups of a result of Levi. For completeness we give the (direct) proof.

Lemma 2 (cf. [Lev] Th. 2 and Cor. 4). *For X a normed group, if (X, d_S) is Polish, i.e. separable and topologically complete, or more generally analytic, and (X, d_R) non-meagre, there is a subset Y of X which is a dense absolute-$\mathcal{G}_\delta$ in (X, d_R), and on which the d_S and d_R topologies agree.*

Proof. As before (X, d_R) is analytic if (X, d_R) is analytic (but see the closing remarks), and being non-meagre is Baire, by Theorem I of Section 3. As (X, d_R) is Baire, the conclusion is implied by an argument of Levi, as follows. Let $\mathcal{B} = \{B_n\}$ be a basis in (X, d_S). Now $j(B_n)$ being analytic in d^X_R has the Baire property (by LSN, §1). So $B_n = (V_n \backslash N_n) \cup M_n$ for some d^X_R-open set V_n and d^X_R-meagre sets N_n, M_n. Without loss of generality for what follows we may suppose that N_n and M_n are $\mathcal{F}_\sigma$ sets. Put $Y := X \backslash \bigcup_n (N_n \cup M_n)$, which is a dense $\mathcal{G}_\delta$ in (X, d_R), a Baire space. Then $B_n \cap Y = V_n \cap Y$ is open both in (Y, d_R) and (Y, d_S). As $\mathcal{B}$ is a basis for (X, d_S), every open set in (Y, d_S) is open in (Y, d_R). Every open set in (Y, d_R) is open in (Y, d_S), since d_S is a refinement of d_R. Thus the two topologies agree on the $\mathcal{G}_\delta$ subset Y. As Y is a $\mathcal{G}_\delta$ subset of (X, d_R), it is also a $\mathcal{G}_\delta$ subset in the complete space (X, d_S), and so (Y, d_S) is topologically complete. So too is (Y, d_R), being homeomorphic to (Y, d_S). Working in Y, we have $y_n \to_R y$ iff $y_n \to_F y$ iff $y_n \to_L y$. ■

Observe that above, since $X \backslash Y$ is meagre under d_R, the space (X, d_R) is almost complete (see Section 3). We use almost completeness to extract much more.

Lemma 3 *If in the setting of Lemma 2 the three topologies generated by d_R, d_L, d_S agree on a dense absolutely-$\mathcal{G}_\delta$ set Y of (X, d_R), then for any $\tau \in Y$ the conjugacy $\gamma_\tau(x) := \tau x \tau^{-1}$ is continuous.*

Proof. We work in (X, d_R). Let $\tau \in Y$. We first establish the continuity in X at e of the conjugacy $x \to \tau^{-1} x \tau$ (by shifting into Y). Let $z_n \to e$

be any null sequence in X. Fix $\varepsilon > 0$; then $T := Y \cap B_\varepsilon^L(\tau)$ is analytic, since T is d_R-open in Y, and is non-meagre, as X is Baire. By Theorem IV of §3 below there is $t \in T$ and t_n in T with t_n converging to t (in d_R, so also in d_L) and an infinite $\mathbb{M}_t$ such that $\{tt_m^{-1}z_m t_m : m \in \mathbb{M}_t\} \subseteq T$. Since the three topologies agree on Y and as the subsequence $tt_m^{-1}z_m t_m$ lies in Y and converges to t in Y under d_R, it also converges to t under d_L. Using the identity $d_L(tt_m^{-1}z_m t_m, t) = d_L(t_m^{-1}z_m t_m, e) = d_L(z_m t_m, t_m)$, we note that

$$\begin{aligned} ||t^{-1}z_m t|| &= d_L(t, z_m t) \leq d_L(t, t_m) + d_L(t_m, z_m t_m) + d_L(z_m t_m, z_m t) \\ &\leq d_L(t, t_m) + d_L(tt_m^{-1}z_m t_m, t) + d_L(t_m, t) \to 0, \end{aligned}$$

as $m \to \infty$ through $\mathbb{M}_t$. So $d_L(t, z_m t) < \varepsilon$ for large enough $m \in \mathbb{M}_t$. Then, as $d_L(\tau, t) < \varepsilon$, for any such m one has

$$\begin{aligned} ||\tau^{-1}z_m \tau|| &= d_L(z_m \tau, \tau) \leq d_L(z_m \tau, z_m t) + d_L(z_m t, t) + d_L(t, \tau) \\ &\leq d_L(\tau, t) + d_L(t, z_m t) + d_L(t, \tau) \leq 3\varepsilon. \end{aligned}$$

Thus for any $\varepsilon > 0$ and any k there is $m = m(k, \varepsilon) > k$ with $||\tau^{-1}z_m \tau|| \leq 3\varepsilon$. Inductively, taking successively $\varepsilon = 1/n$ and $k(n) := m(\varepsilon, k(n-1))$, one has $||\tau^{-1}z_{k(n)}\tau|| \to 0$. By the weak continuity criterion (Lemma 3.5 of [BOst-N], p. 37), $\gamma(x) := \tau^{-1}x\tau$ is continuous. Since (X, d_R^X) is analytic and metric, each open set U is analytic, so $\gamma_\tau^{-1}(U) = \gamma(U)$ is analytic, so has the Baire property by LSN. So $\gamma_\tau(x) = \tau x \tau^{-1} = \gamma^{-1}(x)$ is a Baire homomorphism, and so is continouous – by the Baire Homomorphism Theorem (Th. III of §3). ■

Proof of Theorem 1. Under d_R, the set $Z_\Gamma := \{x : \gamma_x \text{ is continuous}\}$ is a closed subsemigroup of X ([BOst-N], Prop. 3.43). By Lemmas 2 and 3, $X = \mathrm{cl}_R Y \subseteq Z_\Gamma$, i.e. γ_x is continuous for all x, and so (X, d_R^X) is a topological group. So $x_n \to_R x$ iff $x_n^{-1} \to_R x^{-1}$ iff $x_n \to_L x$ iff $x_n \to_S x$. So, being homeomorphic to (X, d_S^X), (X, d_R^X) is a Polish topological group. □

Corollary. *If (X, d_S) is proper, then (X, d_R) is its homeomorph and is a Polish topological group.*

Proof. If d_S is proper, the space (X, d_S) is locally compact and separable, hence topologically complete. So X is semi-Polish. ■

For a further corollary see §4.1.

3 Background on normed groups

We recall four results needed in this paper all but Theorem II were established from the companion paper [Ost-LBIII] ('analytic' is defined in §1). For the notation (especially, $d_R^X(x,y) := ||xy^{-1}||$ and the associated convergence $\to_R$), see Section 2.

Theorem I ([Ost-LBIII] Th. 1)**.** *In a normed group $X, ||.||$ under the right norm topology, i.e. generated by the right invariant metric $d_R^X(x,y) = ||xy^{-1}|$, if X contains a non-meagre analytic set, then X is Baire.*

Theorem II (Equivalence Theorem, [BOst-N] Th. 3.4)**.** *A normed group is a topological group under the right (resp. left) norm topology iff each conjugacy*

$$\gamma_g(x) := gxg^{-1}$$

is right-to-right (resp. left-to-left) continuous at $x = e$ (and so everywhere), i.e. for $z_n \to_R e$ and any g,

$$gz_ng^{-1} \to_R e. \tag{adm}$$

Equivalently, it is a topological group iff left/right-shifts are continuous for the right/left norm topology, or iff the two norm topologies are themselves equivalent.

Theorem III (Baire Homomorphism Theorem, [Ost-LBIII] Th. 4; cf. [Jay-Rog] §2.10, [BOst-N] Th. 11.11)**.** *Let X and Y be normed groups analytic in the right-norm topology with X non-meagre. If $f : X \to Y$ is a Baire homomorphism, then f is continuous.*

Theorem IV (Analytic Shift Theorem, [Ost-LBIII] Th. 2)**.** *In a normed group under the topology d_R^X, with $z_n \to e_X$, A analytic and non-meagre: for a non-meagre set of $t \in A$ with co-meagre Baire envelope, there is an infinite set $\mathbb{M}_t$ and points $a_n \in A$ converging to t such that*

$$\{ta_m^{-1}z_ma_m : m \in \mathbb{M}_t\} \subseteq A.$$

In particular, if the normed group is topological, for quasi all $t \in A$ there is an infinite set $\mathbb{M}_t$ such that

$$\{tz_m : m \in \mathbb{M}_t\} \subseteq A.$$

Remark. Note that $aa_m^{-1}z_ma_m$ converges under d_R to a, as

$$d_R(aa_m^{-1}z_ma_m, a) = ||aa_m^{-1}z_ma_ma^{-1}|| \leq ||aa_m^{-1}|| + ||z_m|| + ||a_ma^{-1}||.$$

The theorem uses shifted-conjugacies to embed a subsequence of the 'null sequence' $z_n \to e_X$ into A; it is natural, borrowing from [?], to term this a 'shift-compactness' – see [?] for background and connections with allied notions of generic automorphisms, and [Ost-S] for a survey of its uses.

4 Concluding remarks

1. *Examples of normed groups.* Some standard examples are provided by subgroups of $Auth(X)$, the algebraic group of self-homeomorphisms (auto-homeomorphisms) of a metric space (X, d^X), for d^X an arbitrary metric, under composition (following the notation of [BePe]). Say that $x \to t(x)$ is *bounded* if $||t|| := \hat{d}(t, id_X) < \infty$, where $id_X(x) \equiv x$ is the identity mapping of X and

$$\hat{d}(t, t') := \sup_x d^X(t(x), t'(x)) \qquad \text{(sup)}$$

denotes the supremum metric (for which see [Eng-2] §4.2). We denote by $\mathcal{H}(X)$ the subgroup of bounded elements of $Auth(X)$. Unless otherwise stated $\mathcal{H}(X)$ is understood to be equipped with the metric $\hat{d}$, which is right-invariant, so that $\mathcal{H}(X)$ is a normed group. The corresponding symmetrization metric is

$$\widetilde{d}(s, t) := \max\{\hat{d}(s, t), \hat{d}(s^{-1}, t^{-1})\},$$

and $\mathcal{H}(X)$ under $\widetilde{d}$ is complete if d^X is complete ([Dug], Th. XIV.2.6, p. 296). (See also [Eng-2] Th. 4.2.16; cf. [vM] Cor. 1.2.16 for possible extensions to locally compact spaces for a related metric.) The subgroup $\mathcal{H}_u(X)$ of $\mathcal{H}(X)$ comprises the homeomorphisms h that are uniformly continuous under d^X and have uniformly continuous inverse h^{-1}. This subgroup is not only complete under $\widetilde{d}$, but also a topological group under $\hat{d}$, for which see [BOst-N, Th. 3.13] or [Dieu].

This context provides the further corollary promised at the end of §2.1.

Proposition 1. *The metric $\hat{d}$ is right-invariant and $\mathcal{H}(X)$ is a normed group under $||h|| := \hat{d}(h, id_X)$. For (X, d^X) complete $\widetilde{d}$ is complete, and so $\mathcal{H}(X)$ is the continuous image of a complete metric space. If additionally under $\hat{d}$ it is separable and non-meagre, then it is a Baire space and a topological group.*
If $\widetilde{d}$ is locally compact, then the topologies of $\widetilde{d}$ and $\hat{d}$ coincide; if $\widetilde{d}$ is separable, for instance for (X, d^X) compact, then $\mathcal{H}(X)$ is analytic (in fact descriptive Borel), and the two topologies agree on a dense $\mathcal{G}_\delta$ of $\mathcal{H}(X)$.

Proof. If $\mathcal{H}(X)$ is separable under $\hat{d}$, then it is separable under $\widetilde{d}$, so we may apply Theorems 1 and 2. ■

For comparison with the uniform topology, note that for X metric $C(X, \mathbb{R})$ under the compact-open topology is separable iff X is locally compact. [Eng-2] ex 3.4E.

2. *Normed groups are topological or pathological.* Evidently (X, d_R^X) and (X, d_L^X) are isometric, via the inversion $i(x) := x^{-1}$. But this does not say that the two metrics are in any way comparable. A slight amount of regularity in the relationship between the left and right norm topologies often in the presence of some topological completeness such as the analyticity of X under d_R^X draws the two into coincidence. Straightforward instances (for which see [Ost-LBIII]) are:
(i) if the graph of the self-homeomorphism $x \to x^{-1}$ is analytic;
(ii) if all the conjugacies $\gamma_t(x) = txt^{-1}$ are Baire under d_R^X;
(iii) if X is locally compact and all the conjugacies $\gamma_t(x) = txt^{-1}$ are Haar-measurable;
(iv) if the norm has the property that there exists a sequence of constants $\kappa_n \to \infty$ such that $\kappa_n ||x|| \leq ||x^n||$ for each $n \in \mathbb{N}$ and $x \in X$ ([BOst-N], Th. 3.39, where the normed group is said to be *Darboux-normed*);
(v) Of course, if X is abelian the two topologies coincide and are admissible (immediate from Theorem II of Section 3).
The result (ii) is connected with the Cauchy dichotomy governing automatic continuuity of homomorphisms. More subtle connections, based on conjugacy, can be formulated in terms of the behaviour of the group's oscillation function on a dense subspace, for which see [BOst-N]. Compare also the density condition (dEV) below.

3. *The Loy and Hoffmann-Jørgensen Theorem.* In the metric case, this result (cited after Theorem 1 in §1) straightforwardly follows from the Steinhaus Subgroup Theorem (for background on this see [BOst-StOstr]): an analytic topological group H may be densely embedded (by completion) in a complete separable topological group G, but now H is a non-meagre subgroup with the Baire property (being analytic), so is all of G. By contrast, however, if a normed group can be extended to a complete normed group, then it is necessarily a topological group (cf. [BOst-N] Th. 3.38).

4. *Normed groups and the Effros Theorem.* The following result was proved by van Mill ([vM]) for T an analytic *topological* group; his proof in fact gives:

Analytic Effros Open Mapping Principle. *For T an analytic* ***normed group*** *acting transitively and separately continuously on a separable metrizable space X: if X is non-meagre, then T acts micro-transitively on X.* Thus the normed group setting is the ideal vehicle for conveying the Effros Principle in this sharp form. For further improvements see [Ost-E].

5. *Proper norms and proper maps.* Under the appropriate circumstances, the map $z \to ||z||$, and so also for any $x \in X$ the map $f_x : z \to d_R^X(z,x)$, is continuous, closed and has inverse images of compact sets compact, i.e. is 'perfect' (or proper), so permitting an embedding of X in the product $\mathbb{R}^X$ via $z \to \langle f_x(z) : x \in X \rangle$ – see [Eng-2] §3.7, [Dug] XI.5, and for analytic applications [Jay-Rog] §5.2; cf. [BH] Remark 3.9.

If d_R^X is proper, then $\bar{B}_r := \{x : ||x|| \le r\}$ is compact under d_R. But $i : x \to x^{-1}$, as a map from (X, d_R^X) to (X, d_L^X), is a homeomorphism which fixes $\bar{B}_r$, so $\bar{B}_r$ is compact under d_L^X and so d_L^X is proper. Thus: *d_R^X is proper iff d_R^X is proper iff the norm is proper.*

For a proper norm, both the left and right norm topologies are locally compact and σ-compact (and so the space is second countable). The norm attains a finite supremum iff the space is compact. If the norm has a finite unattained supremum, w.l.o.g. 1 say, a topologically equivalent unbounded norm is given by $|x| := ||x||/(1 - ||x||)$. Indeed, $d(x,y) := d_R^X(x,y)/(1 - d_R^X(x,y))$ is a right-invariant metric, and $d_R^X(x,y) := d(x,y)/(1 + d(x,y))$ is an equivalent metric (see [Eng-2] Ex. 41.1.B).

A Hausdorff space has a proper metric iff it is locally compact and second countable (a result due to H. E. Vaughan, for which see [Bus1] Th.1.21, where the metrization in the non-compact case is derived from a metrization of a one-point compactification). Compare also [SeKu] §7.3.

6. *The dense Engelking-Vainstein condition (dEV).* This comment is inpired by Vainstein's Theorem that closed maps between metrizable spaces preserve topological completeness, for which see the next Remark and the recent paper [HP]. Let d_S be complete, but not necessarily separable. Suppose that d_L satisfies the following density version (dEV) of a condition, due to Vainstein and studied by Engelking [Eng-1]: for each $\varepsilon > 0$, on a dense set Y of points y there is $\delta > 0$ with $\delta < \varepsilon$ such that $B_R(y, \delta)$ does not contain an infinite subset ε-separated under d_L. (Compare [Eng-2] Th. 4.4.16.) Notice that this implies the existence of two distinct points x, x' near $y \in Y$ under d_R for which $d_L(x, x') < \varepsilon$. The ε-separation condition reappears in [IRSW].

Taking $\varepsilon = 1/n$ for $n \in \mathbb{N}$, and putting $W_n := \{y : (\exists\delta > 0)(\forall K \subseteq B_R(y,\delta))[$ if $d_L(k,k') \geq 1/n$ for distinct $k,k' \in K$, then K is finite$]\}$ is dense open under d_R^X. So if (X, d_R) is again a Baire space, then $H := \bigcap_n W_n$ is a dense $\mathcal{G}_\delta$. By [Eng-1] Lemma 3, on H the continuous embedding map $j(x) = x$ from (X, d_S) to (X, d_R) is closed, i.e. for each $y \in H$ and every d_S-open W with $y \in W$ there is a d_R-open V containing y with $V \subseteq W$. That is, the d_S and d_R topologies agree on H (since d_S refines d_R). This conclusion replaces the preliminary step based on the Levi Lemma in the argument of Section 3 above, and so Theorem 1 also holds under the assumption that (X, d_S) is merely complete, with (X, d_R) Baire (e.g. contains a non-meagre analytic subset), provided d_L satisfies the (dEV) condition; that is:

Theorem 1′ (A Semi-Complete Theorem). *For a normed group X under d_R^X, if the space X is Baire, semi-complete and satisfies the condition (dEV), then it is a Polish topological group.*

7. *Non-separable analogues.* Key to the proof of Theorem 1 is that a continuous image of a complete separable metric space is an analytic space. In the non-separable context continuity is not enough to preserve analyticity, and an additional property is needed to guarantee analyticity, involving σ-discreteness. (See [St2] and [Han-98] Example 4.2 for a non-analytic metric space that is a one-to-one continuous image of κ^ω for some uncountable κ.) We study this matter in [Ost-AB]. Recent work by Holický and Pol ([HP]), in response to Ostrovsky's recent insights, connects preservation of (topological) completeness under continuous maps between metric spaces to the classic notion of *resolvable* sets (for which see [Kur-1] §12 II and V). The latter notion provides the natural generalization to Ostrovsky's more special setting. (Recall that S is resolvable if every non-empty (closed) subset F contains a relatively open set G with $G \subseteq S$ or $G \subseteq F \backslash S$.) They find that a map f preserves completeness if it 'resolves countable discrete sets', i.e. for every countable metrically-discrete set C and open nhd V of C there is L with $C \subseteq L \subseteq V$ such that $f(L)$ is resolvable.

Consider the implications for a normed group X, when f is the identity from (X, d_S) to (X, d_R), and $C = \{c_n\}$ is a d_S-discrete set. (So C and C^{-1} are d_R-discrete, a situation contrasting with the (dEV) condition above.)

To obtain the desired resolvability for this f, it is necessary and sufficient for each C as above and each assignment $r : \mathbb{N} \to \mathbb{R}_+$ with $r_n \to 0$ that there exist d_R-resolvable sets $L_n \subseteq B_R(c_n, r_n) \cap B_L(c_n, r_n)$. Since $\{x : d(c^{-1}, x^{-1}) <$

$r\} = \{x : d(c^{-1}, y) < r$ and $y = x^{-1}\}$, this is yet another condition relating inversion to the d_R-topology, via the sets $B_R(c^{-1}, r)^{-1}$. (Note that if $r_n \to 0$, then $B := \bigcup_n \bar{B}_R(c_n, r_n)$ is closed, so for F closed, if F does not meet any $\bar{B}_R(c_n, r_n)$, then $F \backslash \bigcup_n L_n \supseteq F \backslash B$, which is non-empty and relatively open in F.)

In these circumstances, completeness under d_S entails topological completeness under d_R, so that (X, d_R) is Baire as required in Th. 1. On the other hand, since resolvable sets are $\mathcal{F}_\sigma$ (and $\mathcal{G}_\delta$), the mapping $x \to x^{-1}$ is analytic, and so a separable X is a topological group anyway (see Remark 1(i) above).

8. *Relation to completeness.* The big picture here is that analyticity combined with non-meagreness yields almost completeness; and non-meagreness allows one to avoid meagre parts of space where completeness is missing. Recall that the existence of a dense completely metrizable subspace in a classically analytic space is a result that implicitly goes back to Kuratowski – see [Kur-1] IV.2 p. 88, combined with the result, noted above, that a classically analytic set is Baire in the restricted sense – Cor. 1 p. 482). The group context supports a converse – see the companion paper [Ost-LBIII].

A non-meagre analytic set A in a metric space, as above, may be regarded as a subset of its own metric completion $\hat{A}$. Being dense in its completion, A remains non-meagre and analytic in $\hat{A}$. By LSN (§1), A has the Baire property in $\hat{A}$. Working in the complete space $\hat{A}$, writing $A = (U \backslash N) \cup M$ with N, M meagre, and covering N by a countable union of closed nowhere-dense sets F_n, one deduces that A contains $U \backslash \bigcup_n F_n$, a non-meagre $\mathcal{G}_\delta$. By completeness, A contains a non-meagre metrically complete subset. It is this *almost completeness* (for which see [Mich91]) that analyticity bestows.

The arguments have all been been local in character; a metric space X that is locally complete is complete, since it is locally $\mathcal{G}_\delta$ in the completion X^* and so $\mathcal{G}_\delta$, by Montgomery's Localization Theorem (see [Mont1] and [St1] for generalizations). Likewise, if X is locally analytic, then X is locally analytic in X^*, and so locally Souslin-$\mathcal{F}(X^*)$; again by a theorem of Montgomery X is Souslin-$\mathcal{F}(X^*)$ and so analytic (see [Mont1]). See [ChCN] Ex 2.9 for an example of a locally completely-metrizable space that is metacompact, but not Čech-complete.

9. *Compactness preservation from d_S to d_R.* In part (i) of Theorem 2 there is a hidden subtlety. One may argue that, when d_S is proper, the restriction

of the embedding $j:(X,d_S^X)\to(X,d_R^X)$ to $\bar{B}_r$, being continuous, preserves compactness. So $\bar{B}_r:=\{x:||x||\leq r\}$ is compact under d_R and $j_r:=j|\bar{B}_r$ is a homeomorphism. It is immediate that d_R is proper; but then one must justify why j itself is a homeomorphism. One way forward is that $U:=\{x:d_S(a,x)<\varepsilon\}$ is open under d_S in $\bar{B}_r$ for $r=||a||+2\varepsilon$, so is also open under d_R, being a j_r-homeomorph. But this too is a 'bounding proof' – as in (i) above.

References

[AdGMcD1] J. M. Aarts, J. de Groot, and R. H. McDowell, *Cotopology for metrizable spaces*, Duke Math. J. 37 (1970), 291–295.

[BePe] Cz. Bessaga and A. Pełczyński, *Selected topics in infinite-dimensional topology*, PWN, Warszawa, 1975.

[BOst-N] N. H. Bingham and A. J. Ostaszewski, *Normed versus topological groups: dichotomy and duality*, Dissertationes Math., 472 (2010), 138 pp.

[BOst-StOstr] N. H. Bingham and A. J. Ostaszewski, *Dichotomy and infinite combinatorics: the theorems of Steinhaus and Ostrowski*, Math. Proc. Camb. Phil. Soc., 150.1 (2011), 1-22.

[Bir] G. Birkhoff, *A note on topological groups*, Compositio Math. 3 (1936), 427–430.

[Bou1] A. Bouziad, *The Ellis theorem and continuity in groups*, Topology Appl. 50 (1993), no. 1, 73–80.

[Bou1] A. Bouziad, *Every Čech-analytic Baire semitopological group is a topological group,* Proc. Amer. Math. Soc. 124.3 (1996), 953-959.

[BH] M. Bridson and A. Haefliger, *Metric spaces of non-positive curvature.* Grundlehren der Math. Wiss. 319. Springer-Verlag, 1999.

[Bus1] H. Busemann, *Local metric geometry*, Trans. Amer. Math. Soc. 56.1 (1944), 200-274

[Bus2] H. Busemann, *The geometry of geodesics*, Academic Press, 1955.

[ChCN] J. Chaber, M.M. Čoban, K. Nagami, *On monotonic generalizations of Moore spaces, Čech complete spaces and p-spaces*, Fund. Math. 84.2 (1974), 107–119.

[Dal] H. G. Dales, *Banach algebras and automatic continuity*, London Mathematical Society Monographs. New Series, 24. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 2000.

[Dieu] J. Dieudonné, *On topological groups of homeomorphisms*, Amer. J. Math. 70, (1948), 659–680.

[Dug] J. Dugundji, *Topology,* Allyn and Bacon, 1966.

[Ell1] R. Ellis, *Continuity and homeomorphism groups*, Proc. Amer. Math. Soc. 4 (1953), 969-973.

[Ell2] R. Ellis, *A note on the continuity of the inverse*, Proc. Amer. Math. Soc. 8 (1957), 372–373.

[Eng-1] R. Engelking, *Closed mappings in complete metric spaces,* Fund. Math 70 (1971), 103-107.

[Eng-2] R. Engelking, *General Topology*, Heldermann Verlag, Berlin 1989.

[IRSW] G. Itzkowitz, S. Rothman, H. Strassberg, and T. S. Wu, *Characterization of equivalent uniformities in topological groups.* Topology Appl. 47 (1992), no. 1, 9–34.

[Jay-Rog] J. Jayne and C. A. Rogers, *Analytic sets,* Part 1 of [Rog].

[Han-98] R. W. Hansell, *Non-separable analytic metric spaces and quotient maps*, Topol 85(1998), 143-152.

[HJ] J. Hoffmann-Jørgensen, *Automatic continuity*, Section 3 of [THJ].

[HP] P. Holický and R. Pol, *On a question by Alexey Ostrovsky concerning preservation of completeness*, Topology Appl. 157 (2010), 594-596.

[Kak] S. Kakutani, *Über die Metrisation der topologischen Gruppen*, Proc. Imp. Acad. Tokyo 12 (1936) 82-84 (also in *Selected Papers,* Vol. 1 (ed. Robert R. Kallman), Birkhäuser, 1986, 60-62).

[Kech] A. S. Kechris, *Classical descriptive set theory*, Graduate Texts in Mathematics 156, Springer, 1995.

[Kur-1] K. Kuratowski, *Topology,* Vol. I., PWN, Warsaw 1966.

[Lev] S. Levi, *On Baire cosmic spaces*, General topology and its relations to modern analysis and algebra, V (Prague, 1981), 450–454, Sigma Ser. Pure Math., 3, Heldermann, Berlin, 1983.

[Loy] R. J. Loy, *Multilinear mappings and Banach algebras.* J. London Math. Soc. (2) 14.3 (1976), 423–429.

[Mich91] E. Michael, *Almost complete spaces, hypercomplete spaces and related mapping theorems*, Topology Appl. 41 (1991), no. 1-2, 113–130.

[vM] J. van Mill, *A note on the Effros Theorem*, Amer. Math. Monthly 111.9 (2004), 801-806.

[Mont1] D. Montgomery, *Nonseparable metric spaces*, Fund. Math.25 (1935), 527-534.

[Mont2] D. Montgomery, *Continuity in topological groups*, Bull. Amer. Math. Soc. 42 (1936), 879-882.

[Nam] I. Namioka, *Separate and joint continuity,* Pacific J. Math. 51 (1974), 515–531.

[Ost-AB] A. J. Ostaszewski, *Analytic Baire spaces*, Fundamenta Math., to appear.

[Ost-E] A. J. Ostaszewski, *Almost completeness and the Effros Theorem in normed groups,* Topology Proceedings, to appear.

[Ost-S] A. J. Ostaszewski, *Shift-compactness in almost analytic submetrizable Baire groups and spaces*, Topology Proceedings, to appear.

[Ost-LBIII] A.J. Ostaszewski, *Beyond Lebesgue and Baire III: analyticity and shift-compactness*, preprint.

[Rog] C. A. Rogers, J. Jayne, C. Dellacherie, F. Topsøe, J. Hoffmann-Jørgensen, D. A. Martin, A. S. Kechris, A. H. Stone, *Analytic sets,* Academic Press, 1980.

[SeKu] I. E. Segal and R. A. Kunze, *Integrals and operators*, McGraw-Hill, 1968.

[SolSri] S. Solecki and S.M. Srivastava, *Automatic continuity of group operations*, Top. & Apps, 77 (1997), 65-75.

[St1] A. H. Stone, *Kernel constructions and Borel sets*, Trans. Amer. Math. Soc. 107 (1963), 58-70; errata, ibid. 107 1963 558.

[St2] A. H. Stone, *Analytic sets in non-separable spaces*, Part 5 of [Rog].

[THJ] F. Topsøe, J. Hoffmann-Jørgensen, *Analytic spaces and their applications*, Part 3 of [Rog].

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